Essex County Math League

Wednesday, May 25, 2022

Calculus

ESSEX COUNTY MATH LEAGUE
CALCULUS
MAY 25, 2022

DIRECTIONS: You may write on this test. Be sure that your name, subject, and school (including town name) are on your scantron sheet. Mark the answer sheet with dark, careful marks using a #2 pencil. NG means the answer is "Not Given," and is an acceptable answer. Problem #16 is an open ended question and should be written on the scantron sheet separately. This question will only be used in case of a tie.

1) $\lim_{x\to 3}\frac{\sqrt{x^2+24x}-9}{x-3}=\frac{a}{b}$, where $\frac{a}{b}$ is reduced. Find $a+b$. Note: Let $b = 1$ if limit is an integer.

A) 4 B) 5 C) 7 D) 8 E) NG

2) Let $f(x)=x^3-5x^2+7x$ on the interval $[-1,2]$. Let $c=a+b\sqrt{d}$ be the real number between -1 and 2 that satisfies the Mean Value Theorem, where a, b, are rational numbers, and d is a square free integer. Find $2a+b+d$. Note: If c is a rational number, let $b=0$ and $d=1$.

A) 22 B) $\frac{68}{3}$ C) 28 D) $\frac{475}{6}$ E) NG

3) Let $g(x)=\frac{(x^2+a)^6}{(bx+2)^5}$, where a and b are constants. The derivative is $g'(x)=\frac{(21x^2+24x-6)(x^2+a)^5}{(bx+2)^6}$. Find $a-b$.

A) $-\frac{44}{5}$ B) $-\frac{31}{10}$ C) $-\frac{13}{5}$ D) $-\frac{21}{20}$ E) NG

4) Let G be a function such that $G'(x)=e^{x^2}$ and $G(0)=1$. Find $\lim_{x\to 0}\frac{x^2G(x)}{\cos(3x)-1}$.

A) $-\frac{2}{3}$ B) $-\frac{2}{9}$ C) 0 D) Does not exist E) NG

5) Let $f(x)=\begin{cases} ax^3+bx^2+3, & \text{for } x<2 \\ ax^2-b, & \text{for } x\geq 2 \end{cases}$ Determine the values of a and b for which f is both continuous and differentiable everywhere. Find $3a + 4b$.

A) -5 B) -3 C) $-\frac{5}{2}$ D) $\frac{11}{2}$ E) NG

6) $\int_0^2 \frac{x}{4x^4+1}\,dx = a\arctan b$, where a is a rational number and b is an integer. Find $12a - 5b$.

A) -37 B) -34 C) -7 D) -4 E) NG

7) Let f be a function such that $f'(2)=1$. Find $\lim_{h\to 0}\frac{f(2+3h)-f(2+h)}{h}$.

A) 1 B) 2 C) 3 D) 4 E) NG

8) Let R be the region in the first quadrant bounded by $y=\ln x$, $y=x+1$, $y=1$, $y=4$. Find the area of R.

A) $\frac{15}{2} - \ln 256$ B) $\frac{27}{2} - \ln 256$ C) $e^4 - e - \frac{9}{2}$ D) $\frac{1}{2}e^8 - 2e^4$ E) NG

9) Find the x-intercept of the line tangent to the curve $xy^2 = x^3y + 30$ at the point $(3, -1)$.

A) $\frac{117}{28}$ B) $\frac{38}{9}$ C) $\frac{111}{26}$ D) 42 E) NG

10) Let $f(x) = \begin{cases} 4-3x^2, & \text{if } x<1 \\ 2x+5, & \text{if } x \geq 1 \end{cases}$. Find c such that $\int_0^c f(x) = 10$. Write c as $a + b\sqrt{d}$, where a, b, are rational numbers, and d is a square free integer. Find $a + b + d$. Note: If c is a rational number, let $b = 0$ and $d = 1$.

A) 51 B) 63 C) 75 D) 87 E) NG

11) Let $f(x) = \int_1^{\ln x} \sqrt{e^t + 1}\, dt$. Find $f'(2)$.

A) $\frac{\sqrt{3}}{2}$ B) $\sqrt{3}$ C) $\frac{\sqrt{e^2+1}}{2}$ D) $\sqrt{e^2+1}$ E) NG

12) Let $f(x) = \sqrt[3]{x}$. Which of the following statements are true.

I) f is continuous at $x = 0$.
II) f is differentiable at $x = 0$.
III) There is a line tangent to f at $x = 0$.

A) I only B) I,II,III C) I,III only D) III only E) NG

13) Let $g(x) = 5^{x\ln x}$. Find $g'(1)$.

A) 0 B) 1 C) $\ln 5$ D) undefined E) NG

14) Let $f(x) = x + \ln(2x-1)$. Find the maximum slope of the lines tangent to the graph of f on the interval [1,3].

A) 1 B) 3 C) 4 D) no maximum E) NG

15) A spherical balloon is being inflated at a rate of 500 cm^3/min. Find the rate that the radius is changing, in cm/min, when the surface area is 100 cm^2.

A) $\dfrac{1}{800\pi}$ B) $\dfrac{25}{\sqrt[3]{45\pi}}$ C) $\dfrac{25}{2\sqrt{\pi}}$ D) 5 E) NG

TIE BREAKER

16) A ten foot wire is to be divided into two pieces. One piece will form a circle, and the other will form a rectangle where the length is twice the width. Find the length of the wire for the circle that minimizes the sum of the areas of the circle and the rectangle. Approximate answer to the nearest hundredth of a foot.

ECML 2022 Calculus

1. D
2. A
3. C
4. B
5. C
6. C
7. B
8. C
9. A
10. C
11. A
12. C
13. C
14. B
15. D
16. 4.11

ECML MAY 2022
CALCULUS SOLUTIONS

1. $\lim_{x\to 3} \frac{\sqrt{x^2+24x}-9}{x-3} = \lim_{x\to 3} \frac{(\sqrt{x^2+24x}-9)(\sqrt{x^2+24x}+9)}{(x-3)(\sqrt{x^2+24x}+9)} = \lim_{x\to 3} \frac{x^2+24x-81}{(x-3)(\sqrt{x^2+24x}+9)}$

$= \lim_{x\to 3} \frac{(x-3)(x+27)}{(x-3)(\sqrt{x^2+24x}+9)} = \lim_{x\to 3} \frac{x+27}{\sqrt{x^2+24x}+9} = \frac{3+27}{\sqrt{3^2+24\cdot 3}+9} = \frac{30}{\sqrt{81}+9} = \frac{30}{18} = \frac{5}{3}$

$a+b = 5+3 = \boxed{8}$ (D)

2. f is continuous and differentiable on $[-1,2]$ so MVT can be applied.
$f'(x) = 3x^2-10x+7$ and $\frac{f(2)-f(-1)}{2-(-1)} = \frac{2-(-13)}{3} = \frac{15}{3} = 5$. By MVT, there exists c, $-1<c<2$ such that $f'(c) = 3c^2-10c+7 = 5 \to 3c^2-10c+2=0 \to$

$c = \frac{-(-10)\pm\sqrt{(-10)^2-4\cdot 3\cdot 2}}{2\cdot 3} = \frac{10\pm\sqrt{76}}{6} = \frac{10\pm 2\sqrt{19}}{6} = \frac{5\pm\sqrt{19}}{3}$

$\frac{5+\sqrt{19}}{3} > 2$, so $c = \frac{5-\sqrt{19}}{3} = \frac{5}{3} - \frac{1}{3}\sqrt{19}$ $\quad 2a+b+d = 2\left(\frac{5}{3}\right) - \frac{1}{3} + 19 = \boxed{22}$ (A)

3. $g'(x) = \frac{(bx+2)^5\cdot 6(x^2+a)^5\cdot 2x - (x^2+a)^6\cdot 5(bx+2)^4 b}{[(bx+2)^5]^2}$

$= \frac{(bx+2)^4(x^2+a)^5[12x(bx+2)-5b(x^2+a)]}{(bx+2)^{10}} = \frac{(x^2+a)^5(12bx^2+24x-5bx^2-5ab)}{(bx+2)^6}$

$= \frac{(7bx^2+24x-5ab)(x^2+a)^5}{(bx+2)^6} \to \begin{matrix} 7b=21 \\ -5ab=-6 \end{matrix} \to b=3$ and $-15a=-6 \to a = \frac{-6}{-15} = \frac{2}{5}$

$a-b = \frac{2}{5}-3 = \boxed{-\frac{13}{5}}$ (C)

4. $\lim_{x\to 0} \frac{x^2G(x)}{\cos(3x)-1} = \lim_{x\to 0} \frac{x^2e^{x^2}+2xG(x)}{-3\sin(3x)}$ (by L'Hopital's Rule)

form $\frac{0}{0}$ $\qquad$ also form $\frac{0}{0}$

$= \lim_{x\to 0} \frac{2x^3e^{x^2}+2xe^{x^2}+2xe^{x^2}+2G(x)}{-9\cos(3x)} = \frac{0+0+0+2(1)}{-9(1)} = \boxed{-\frac{2}{9}}$ (B)

5. For any a, b, f is continuous and differentiable everywhere except possibly at $x=2$. For f to be continuous and differentiable at $x=2$, need $ax^3+bx^2+3 = ax^2-b$ and $\frac{d}{dx}(ax^3+bx^2+3) = \frac{d}{dx}(ax^2-b) \rightarrow 3ax^2+2bx = 2ax$ at $x=2$

$$\rightarrow \begin{matrix} 8a+4b+3 = 4a-b \\ 12a+4b = 4a \end{matrix} \rightarrow \begin{matrix} 4a+5b=-3 \\ 8a+4b=0 \end{matrix} \rightarrow 4b=-8a \rightarrow b=-2a$$

$\rightarrow 4a-10a=-3 \rightarrow -6a=-3 \rightarrow a=\frac{1}{2}$ and $b=-2(\frac{1}{2})=-1$

$3a+4b = 3(\frac{1}{2})+4(-1) = \boxed{-\frac{5}{2}}$ (C)

6. $\int_0^2 \frac{x}{4x^4+1}dx = \frac{1}{4}\int_0^8 \frac{1}{u^2+1}du = \frac{1}{4}\arctan u\Big|_0^8 = \frac{1}{4}\arctan 8 - \frac{1}{4}\arctan 0$

let $u=2x^2$, $du = 4x\,dx$, $\frac{1}{4}du = x\,dx$; $x=0 \rightarrow u=0$, $x=2 \rightarrow u=8$

$= \frac{1}{4}\arctan 8 - 0 = \frac{1}{4}\arctan 8$

$12a-5b = 12\cdot\frac{1}{4} - 5\cdot 8 = \boxed{-37}$ (A)

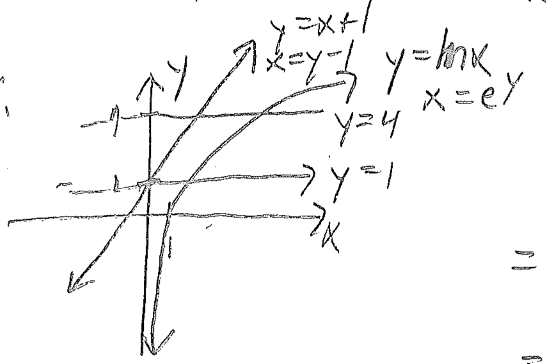
7. $\lim_{h\to 0}\frac{f(2+3h)-f(2+h)}{h} = \lim_{h\to 0}\frac{(f(2+3h)-f(2))-(f(2+h)-f(2))}{h} =$

$\lim_{h\to 0}\frac{f(2+3h)-f(2)}{h} - \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{k\to 0}\frac{f(2+k)-f(2)}{k/3} - f'(2)$

let $k=3h$, as $h\to 0$, $k\to 0$

$= 3\lim_{k\to 0}\frac{f(2+k)-f(2)}{k} - f'(2) = 3f'(2)-f'(2) = 3\cdot 1 - 1 = \boxed{2}$ (B)

8.



$A = \int_1^4 [e^y-(y-1)]dy = \int_1^4 (e^y-y+1)dy$

$= (e^y - \frac{1}{2}y^2+y)\Big|_1^4 = (e^4-\frac{1}{2}\cdot 4^2+4)-(e^1-\frac{1}{2}\cdot 1^2+1)$

$= e^4-4-e-\frac{1}{2} = \boxed{e^4-e-\frac{9}{2}}$ (C)

9. $xy^2 = x^3y+30$

$\rightarrow x\cdot 2y\frac{dy}{dx}+y^2 = x^3\frac{dy}{dx}+3x^2y$

$\rightarrow (2xy-x^3)\frac{dy}{dx} = 3x^2y-y^2$

$\rightarrow \frac{dy}{dx} = \frac{3x^2y-y^2}{2xy-x^3}$

at $(3,-1)$ $\frac{dy}{dx} = \frac{3(3^2)(-1)-(-1)^2}{2(3)(-1)-3^3} = \frac{-28}{-33} = \frac{28}{33}$

$y-(-1) = \frac{28}{33}(x-3) \rightarrow y+1 = \frac{28}{33}x - \frac{28}{11}$

$\rightarrow y = \frac{28}{33}x - \frac{39}{11}$

when $y=0$

$-\frac{28}{33}x = -\frac{39}{11} \rightarrow -28x = -117$

$x = \boxed{\frac{117}{28}}$ (A)

10. f is nonnegative for $x>0$, so $\int_0^x f(t)dt$ is an increasing function, So there is at most one value c, $c>0$ where $\int_0^c f(x)dx = 10$.

Also, $\int_0^1 f(x)dx = \int_0^1 (4-3x^2)dx = (4x-x^3)\Big|_0^1 = (4\cdot 1 - 1^3) - 0 = 3 \rightarrow c>1$

$\int_0^c f(x)dx = 10 \rightarrow \int_0^1 f(x)dx + \int_1^c f(x)dx = 10 \rightarrow 3 + \int_1^c (2x+5)dx = 10$

$\rightarrow \int_1^c (2x+5)dx = 7 \rightarrow (x^2+5x)\Big|_1^c = 7 \rightarrow [(c^2+5c)-(1^2+5\cdot 1)] = 7$

$\rightarrow c^2+5c-6 = 7 \rightarrow c^2+5c-13 = 0 \rightarrow c = \frac{-5 \pm \sqrt{5^2-4(1)(-13)}}{2} = \frac{-5 \pm \sqrt{77}}{2}$

Since $c>1$, $c = -\frac{5}{2} + \frac{1}{2}\sqrt{77}$ $a+b+d = -5+2+77 = \boxed{75}$ (C)

11. Let $g(x) = \int_1^x \sqrt{e^t+1}\,dt$. By Fundamental Theorem of Calculus,

$g'(x) = \sqrt{e^x+1}$. Since $f(x) = g(\ln x)$, by Chain Rule $f'(x) = g'(\ln x)\cdot\frac{d}{dx}(\ln x)$

$= \sqrt{e^{\ln x}+1}\cdot\frac{1}{x} = \frac{\sqrt{x+1}}{x}$, $f'(2) = \frac{\sqrt{2+1}}{2} = \boxed{\frac{\sqrt{3}}{2}}$ (A)

12. Since f is an algebraic function defined everywhere, f is continuous everywhere, including at $x=0$. $f'(x) = \frac{1}{3}x^{-2/3}$, which is not defined at $x=0$. So f is not differentiable at $x=0$. $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^{1/3}-0}{h} = \lim_{h\to 0} h^{-2/3} = \infty$

$\rightarrow$ tangent line (vertical) exists at $x=0$. So $\boxed{I, III}$ are true. (C)

13. $g'(x) = 5^{x\ln x}\cdot\ln 5\cdot\frac{d}{dx}(x\ln x) = 5^{x\ln x}\cdot\ln 5(x\cdot\frac{1}{x}+\ln x) = 5^{x\ln x}\ln 5(1+\ln x)$

$g'(1) = 5^{1\ln 1}\ln 5(1+\ln 1) = 5^0\ln 5\cdot 1 = \boxed{\ln 5}$ (C)

14. $f'(x) = 1+\frac{2}{2x-1}$ is continuous on $[1,3]$. So by Extreme Value Theorem, f' has a maximum value on $[1,3]$, i.e, f has a tangent line with a maximum slope on $[1,3]$ $f'(x) = 0 \rightarrow 1+\frac{2}{2x-1} = 0 \rightarrow \frac{2x-1+2}{2x-1} = 0 \rightarrow \frac{2x+1}{2x-1} = 0$

$\rightarrow 2x+1 = 0 \rightarrow x = -\frac{1}{2}$. Since $-\frac{1}{2}<1$, maximum occurs at an endpoint

$f'(1) = 1+\frac{2}{2(1)-1} = 1+2 = 3$ $f'(3) = 1+\frac{2}{2\cdot 3-1} = 1+\frac{2}{5} = \frac{7}{5}$.

So maximum value is $\boxed{3}$ (B)

15. $\frac{dV}{dt} = 500 \text{ cm}^3/\text{min}$ $V = \frac{4}{3}\pi r^3$ when $A = 100$

$\frac{dr}{dt} = ?$ when $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\frac{dr}{dt} = \boxed{5}$ cm/min

surface area $= 100 \text{ cm}^2$ when $A = 100$ (D)

$A = 4\pi r^2$ $500 = 100 \frac{dr}{dt}$

16. Let x = amount of wire for circle, then $10 - x$ = amount of wire for rectangle

Circle

$C = 2\pi r$

$x = 2\pi r$

$\frac{x}{2\pi} = r$

$A_C = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2$

$= \frac{\pi x^2}{4\pi^2} = \frac{1}{4\pi}x^2$

Rectangle

$P = 2l + 2w = 2(2w) + 2w = 6w$

$10 - x = 6w$

$w = \frac{10-x}{6}$

$A_R = lw = (2w)w = 2w^2$

$= 2\cdot\left(\frac{10-x}{6}\right)^2$

$= 2\cdot\frac{100-20x+x^2}{36} = \frac{x^2-20x+100}{18}$

$= \frac{1}{18}x^2 - \frac{10}{9}x + \frac{50}{9}$

Total Area, $A = \frac{1}{4\pi}x^2 + \frac{1}{18}x^2 - \frac{10}{9}x + \frac{50}{9}$

$= \left(\frac{1}{4\pi} + \frac{1}{18}\right)x^2 - \frac{10}{9}x + \frac{50}{9} = \frac{2\pi+9}{36\pi}x^2 - \frac{10}{9}x + \frac{50}{9}$

A is an upwards parabola, so minimum occurs at vertex provided x-coordinate of vertex is between 0 and 10

$A'(x) = \frac{2\pi+9}{18\pi}x - \frac{10}{9} = 0 \rightarrow \frac{2\pi+9}{18\pi}x = \frac{10}{9}$

$\rightarrow x = \frac{10}{9}\cdot\frac{18\pi}{2\pi+9} = \frac{20\pi}{2\pi+9}$

$0 < \frac{20\pi}{2\pi+9} < 10$, so minimum occurs at $\frac{20\pi}{2\pi+9} \approx$ (4.11) ft.